

Lecture 9 Rigid Body II

- moment of inertia
 - parallel axis theorem
 - cross product
 - torque and Newton's 2nd Law in rotational dynamics.
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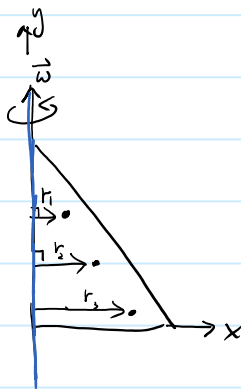
Moment of inertia

Recall for a rigid body rotating about a fixed axis, the kinetic energy is

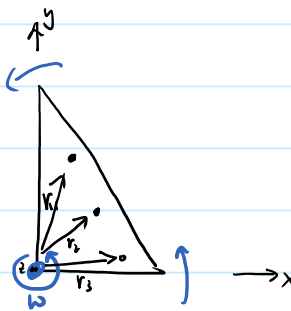
$$K = \frac{1}{2} I \omega^2$$

where $I = \sum_i m_i r_i^2$ ← moment of inertia ("laziness" in rotation)

r_i is the perpendicular distance from the rotation axis to the mass m_i .



rotating about y-axis

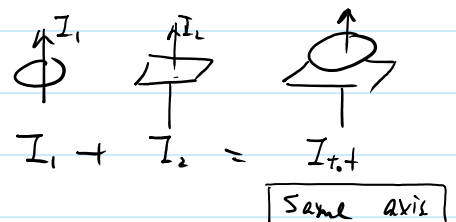


rotating about z-axis.

Properties of I : - Shape dependent. (not just mass)

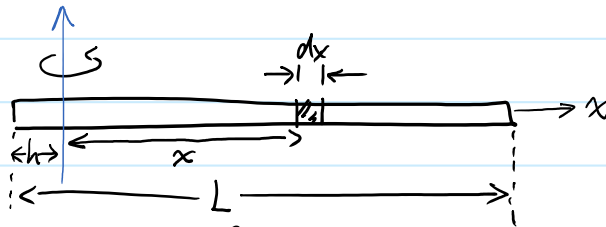
- Axis dependent. (r_i depends on the rotation axis)

- Additive $I = I_1 + I_2$



Example - Rod with uniform linear density $\rho = \frac{M}{L}$.

Rod.



total mass = M .

mass per unit length = $\frac{M}{L}$

$$I = \sum m_i r_i^2 \rightarrow \int dm \cdot x^2$$

$$\text{Eq: } dm = \rho dx \\ = \frac{M}{L} \cdot dx$$

$$I = \int_{-h}^{L-h} \frac{M}{L} dx \cdot x^2$$

$$= \frac{M}{L} \frac{1}{3} x^3 \Big|_{-h}^{L-h}$$

$$= \frac{M}{3L} [(L-h)^3 - (-h)^3]$$

$$(L-h)^3 = L^3 - 3L^2h \\ + 3Lh^2 - h^3$$

$$= \frac{M}{3} (L^2 - 3Lh + 3h^2)$$

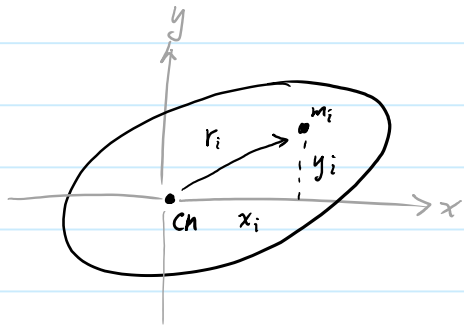
- Rotating about the center of the rod

$$h = L/2 \Rightarrow I = I_{cm} = \frac{M}{3} \left(1 - \frac{3}{2} + \frac{3}{4} \right) L^2 \\ = \frac{1}{12} ML^2$$

- Rotating about the end of the rod.

$$h = 0 \Rightarrow I_{end} = \frac{1}{3} ML^2$$

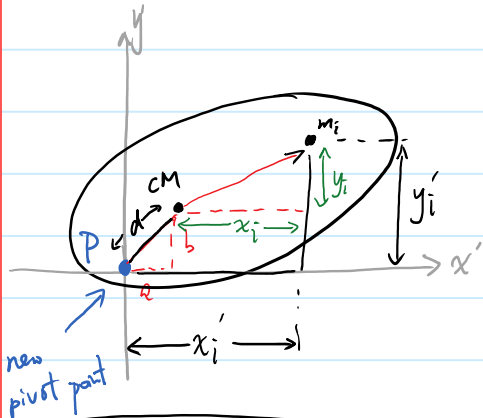
Parallel axis theorem.



CM is at $(x_{cm}, y_{cm}) = \left(\frac{1}{M_{tot}} \sum_i m_i x_i, \frac{1}{M_{tot}} \sum_i m_i y_i \right) = (0, 0)$ ^{at origin}

Moment of inertia about the CM.
$$I_{cm} = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2)$$

Now we want to rotate the object at P.
and define a new coordinate system $x'-y'$ plane.
in where the CM is located at (a, b) .



Moment of inertia about P.

$$I_p = \sum_i m_i (x_i'^2 + y_i'^2)$$

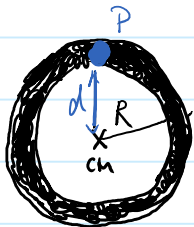
$$\begin{aligned} &= \sum_i m_i [(a + x_i)^2 + (b + y_i)^2] \\ &= \sum_i m_i (a^2 + b^2) + \sum_i m_i (x_i^2 + y_i^2) + 2 \left(\sum_i m_i x_i \right) a + 2 \left(\sum_i m_i y_i \right) b \end{aligned}$$

$\sum_i m_i x_i = M_{tot} x_{cm}$ but $x_{cm} = 0$ so as y_{cm}

$$\Rightarrow \sum_i m_i x_i = 0, \quad \sum_i m_i y_i = 0$$

$$\Rightarrow I_p = I_{cm} + M_{tot} \cdot d^2 \quad \text{parallel axis theorem.}$$

Example. Ring . mass = M

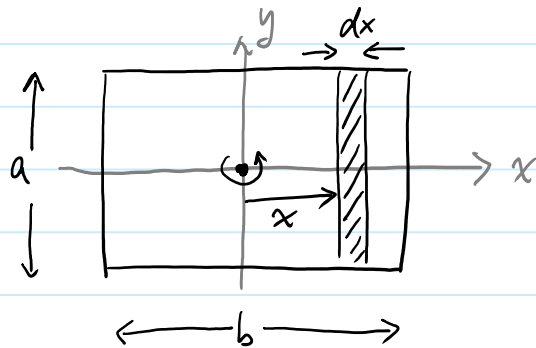
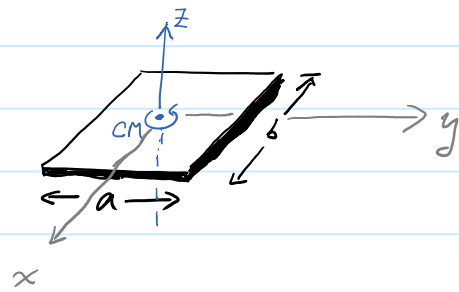


$$I_{cm} = \sum_i m_i r_i^2 = \sum_i m_i R^2 = M R^2$$

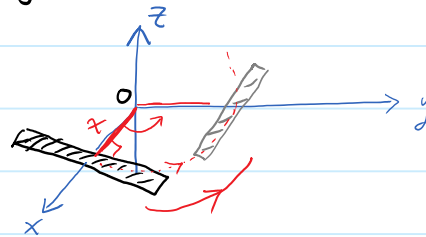
$$\begin{aligned} I_p &= I_{cm} + M \cdot d^2 & d &= R \\ &= M R^2 + M R^2 \\ &= 2 M R^2 \end{aligned}$$

harder to rotate a ring about P

Example. Rectangular plate.
about z-axis through its CM



Consider the mass element dm as a rod at $\perp$ distance x from the z-axis.



$$dI_{cm} = \frac{1}{12} dm \cdot a^2$$

using parallel axis theorem

$$\begin{aligned} dI_o &= dI_{cm} + dm \cdot x^2 \\ &= \left(\frac{a^2}{12} + x^2 \right) dm \end{aligned}$$

$$\text{where } dm = \frac{M}{b} \cdot dx$$

Moment of inertia of the plate : Sum of all dI_o

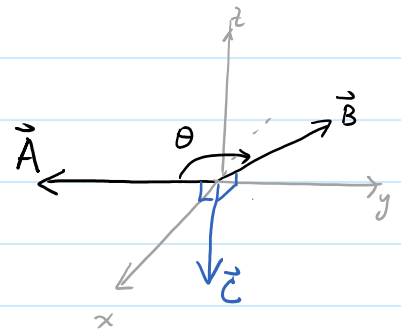
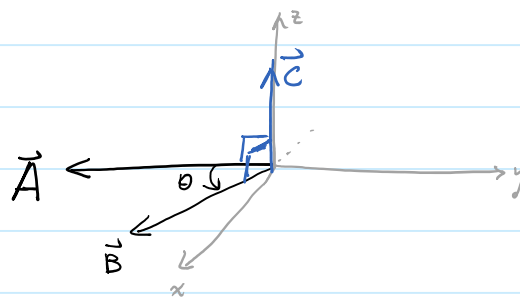
$$I = \int_{x=-b/2}^{x=b/2} dI_o = \frac{M}{b} \int_{-b/2}^{b/2} \left(\frac{a^2}{12} + x^2 \right) dx$$

$$= \frac{M}{b} \left[\frac{a^2}{12} b + \frac{1}{3} \left(\frac{b}{2} \right)^3 \cdot 2 \right]$$

$$= \frac{1}{12} M(a^2 + b^2)$$

Cross Product.

$$\vec{C} = \vec{A} \times \vec{B}$$



magnitude: $|\vec{C}| = A \cdot B \cdot \sin \theta$

direction: using Right Hand Rule. , $\vec{C} \perp \vec{B}$ & $\vec{C} \perp \vec{A}$.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \text{anti-commutative.}$$

Special cases: If $\vec{A} \parallel \vec{B}$, $\vec{A} \times \vec{B} = \vec{0}$. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

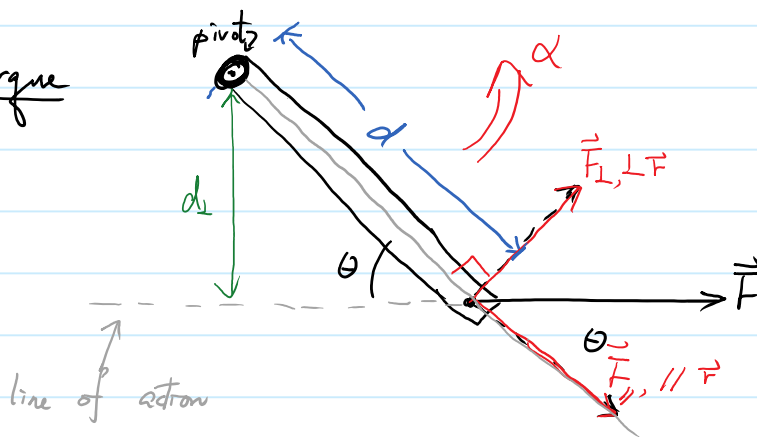
If $\vec{A} \perp \vec{B}$, $|\vec{A} \times \vec{B}| = A \cdot B$.

$$\text{RHR: } \begin{cases} \hat{i} \times \hat{j} = \hat{k} \\ \hat{j} \times \hat{k} = \hat{i} \\ \hat{k} \times \hat{i} = \hat{j} \end{cases}$$

In general,

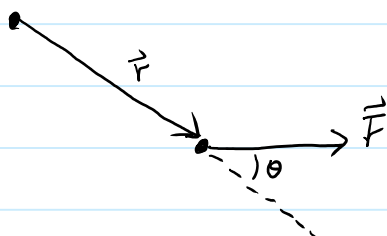
$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

Torque



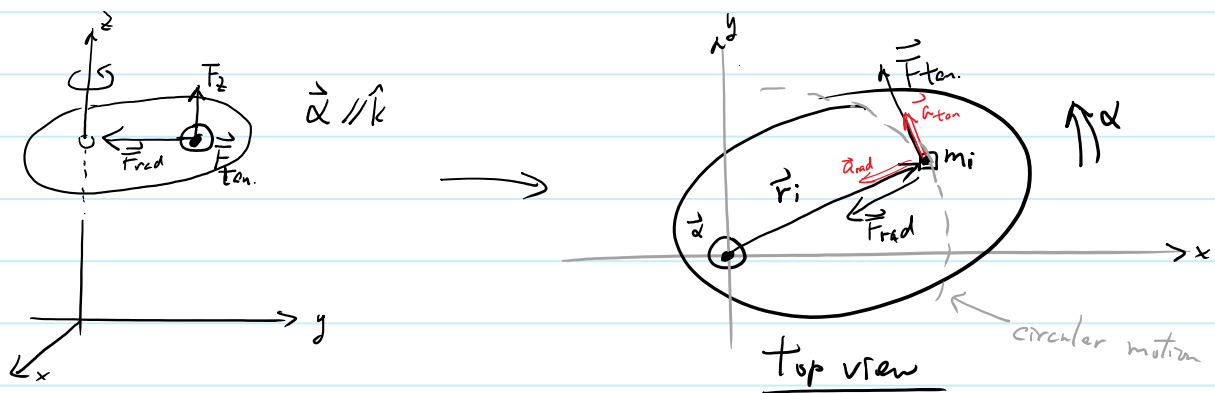
Torque $\propto d$, $F_{\perp}$ only.

$$\begin{aligned} \tau &= d \cdot F_{\perp} \\ &= d \cdot F \cdot \sin \theta \\ &= F \cdot d_{\perp} \end{aligned}$$



$$\vec{\tau} = \vec{r} \times \vec{F} \quad \odot$$

$\vec{\tau}$ points out of the page.



for every mass in the object,

tangential component of $\vec{a}$ $a_{i,tan} = r_i \alpha$

Newton's 2nd Law.

$$m_i a_{i,tan} = F_{i,tan}$$

$$r_i \times m_i r_i \alpha = F_{i,tan} \times r_i$$

$$m_i r_i^2 \alpha = r_i F_{i,tan}$$

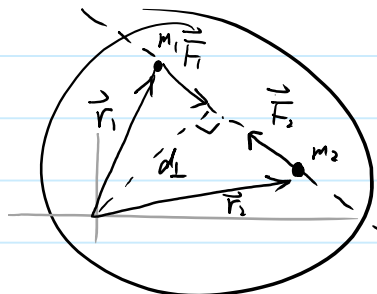
$$I_{i,z} \alpha = \tau_{i,z}$$

$$\Rightarrow \sum_i \tau_{i,z} = \sum_i I_{i,z} \alpha$$

$$\Rightarrow \boxed{\tau_{tot} = \tau_{ext} = I_z \alpha}$$

↑
Only external forces are included.

What about torques due to internal forces?



$\vec{\tau}_1 \otimes$ into the page
 $\vec{\tau}_2 \odot$ out of the page } opposite in direction

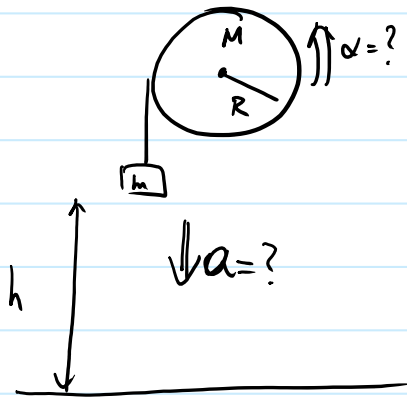
$$|\vec{\tau}_1| = F_1 \cdot d_1$$

$$|\vec{\tau}_2| = F_2 \cdot d_2$$

$$|\vec{F}_1| = |\vec{F}_2| \Rightarrow |\vec{\tau}_1| = |\vec{\tau}_2|$$

$$\Rightarrow \vec{\tau}_1 = -\vec{\tau}_2, \quad \sum \vec{\tau}_{int} = \vec{\tau}_1 + \vec{\tau}_2 = \vec{0}$$

Review



$$\textcircled{1} \quad L_z = I_z \alpha_z$$

$$\textcircled{2} \quad \vec{F} = m \vec{a}$$

$$\textcircled{3} \quad v_{\text{tan}} = r\omega \Rightarrow a_{\text{tan}} = r\alpha.$$

For the cylinder, $\odot z$ as +ve.

$$L_z = I_z \alpha_z$$

$$+RT = I_{\text{cm}} \alpha$$

$$T = I_{\text{cm}} \frac{\alpha}{R}$$

$$\alpha = \frac{a}{R}$$

$$T = \frac{\frac{1}{2}MR^2 a}{R^2}$$

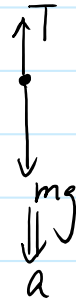
$$\Rightarrow T = \frac{1}{2}Ma$$

For the mass, $F_{\text{net}} = ma$

$$mg - T = ma$$

$$mg - \frac{1}{2}Ma = ma$$

$$a = \frac{mg}{m + \frac{1}{2}M} = \frac{g}{1 + \frac{M}{2m}} < g.$$

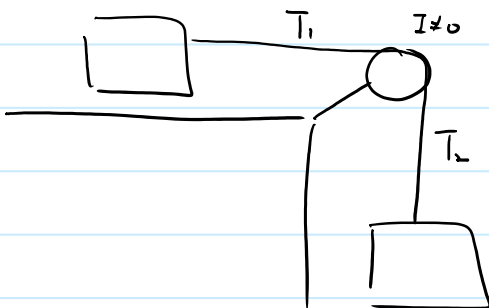


check. $v_f^2 = v_i^2 + 2ah$

$$v_f = \sqrt{\frac{2gh}{1 + \frac{M}{2m}}} = \sqrt{\frac{2mgh}{m + \frac{I_{\text{cm}}}{R^2}}}$$

$$I_{\text{cm}} = \frac{1}{2}MR^2$$

$$\frac{1}{2}M =$$



$$T_1 \neq T_2$$

$$T_2 > T_1$$

$$T_{\text{net}} = (T_2 - T_1)R > 0$$

for non-zero α .

